BEAM EXTRACTION AT A THIRD INTEGRAL RESONANCE III

K. R. Symon

April 29, 1968

In this report III of this series, we will carry the development of report II to one further level of approximation by carrying out the transformation which removes the neglected terms in the Hamiltonian H_3. In this way, we obtain correction terms to the variables X, P used in report I, and defined by Eqs (II-13), (II-17), and (II-25). Thus we can estimate, for example, deviations of the actual phase plot from the idealized plot of Fig. I-1. We also obtain contributions of the neglected terms to H_4, which will enable us to calculate the bending of the separatrices in Fig. I-1.

Consider a typical non-resonant term

$$H_{3\ell m}\,(2\rho)^{3/2}\cos\,(\ell\gamma - m\Theta + \eta_{3\ell m}) \tag{1}$$

in the Hamiltonian (II-21) where $\ell = 1$ or 3. The final approximate Hamiltonian (II-26) was obtained by neglecting all terms (1) except the resonant term $\ell = 3$, $m = m_0$ which drives the extraction resonance $\nu = m_0/3$.

To eliminate the term (1), we utilize the generating function

$$S = \rho'\gamma + S_{3\ell m}\,(2\rho')^{3/2}\sin\,(\ell\gamma - m\Theta + \eta_{3\ell m}), \tag{2}$$

which gives the transformation

$$\rho = \frac{\partial S}{\partial \gamma} = \rho' + \ell S_{3\ell m} (2\rho')^{3/2} \cos (\ell\gamma - m\Theta + \eta_{3\ell m}),$$

$$\gamma' = \frac{\partial S}{\partial \rho'} = \gamma + 3S_{3\ell m}(2\rho')^{1/2} \sin (\ell\gamma - m\Theta + \eta_{3\ell m}). \tag{3}$$

The latter equation can be by approximations for

$$\gamma = \gamma' - 3S_{3\ell m} (2\rho')^{1/2} \sin (\ell\gamma' - m\Theta + \eta_{3\ell m}) + \cdots . \tag{4}$$

We substitute the first of Eqs. (3) in $H = H_2 + H_3 + \cdots$ to obtain

$$\begin{aligned} H' = H + \frac{\partial S}{\partial \Theta} &= \nu\rho' + \cdots \\ &+ \left[H_{3\ell m} + (\ell\nu - m)S_{3\ell m}\right] (2\rho')^{3/2} \cos (\ell\gamma - m\Theta + \eta_{3\ell m}) \\ &+ H_{3\ell m}\left[(2\rho)^{3/2} - (2\rho')^{3/2}\right] \cos (\ell\gamma - m\Theta + \eta_{3\ell m}) + \cdots \end{aligned} \tag{5}$$

The third order term can be eliminated by setting

$$S_{3\ell m} = \frac{H_{3\ell m}}{m - \ell\nu}, \tag{6}$$

provided we are not too close to the resonance $\nu = m/\ell$.

After carrying out the above transformation on every nonresonant term, we arrive at a Hamiltonian H' which to third order contains only the terms in Eq. (II-24). Equation (II-24) is therefore exact to third order provided we replace ρ, γ by the variables

$$\rho' = \rho - \sum_{\ell=1,3} \sum_{m}{}' \frac{\ell H_{3\ell m}}{m - \ell\nu} (2\rho)^{3/2} \cos (\ell\gamma - m\Theta + \eta_{3\ell m}) + \cdots,$$

$$\gamma' = \gamma + \sum_{\ell=1,3} \sum_{m}{}' \frac{3H_{3\ell m}}{m - \ell\nu} (2\rho)^{1/2} \sin (\ell\gamma - m\Theta + \eta_{3\ell m}) + \cdots . \tag{7}$$

The prime on the summation sign means that the resonant term $\ell = 3$, $m = m_0$ is omitted. In rectangular coordinates at the septum ($\Theta = 0$): (8)

$$X = X' - \sum_{m} \frac{H_{31m}}{m - \nu} \left[(3P'^2 + X'^2) \sin \eta_{31m} + 2P'X' \cos \eta_{31m}\right]$$

$$- \sum_{m}{}' \frac{3H_{33m}}{m - 3\nu} \left[(P'^2 - X'^2) \sin \eta_{33m} - 2P'X' \cos \eta_{33m}\right] + \cdots$$

$$P = P' + \sum_{m} \frac{H_{31m}}{m - \nu} \left[(3X'^2 + P'^2) \cos \eta_{31m} + 2P'X' \sin \eta_{31m}\right]$$

$$- \sum_{m}{}' \frac{3H_{33m}}{m - 3\nu} \left[(X'^2 - P'^2) \cos \eta_{33m} + 2P'X' \sin \eta_{33m}\right] + \cdots$$

Equation (II-24), which leads to Eq. (II-26) on which report I is based, should be understood now in terms of the primed variables. In the figure and equations of report I, all capital variables should be **primed**. Equations (II-13) give the connection between the unprimed variables X,P and the original betatron variables x, $dx/d\Theta$, mentioned in the first paragraph of report I. The figures and results of I should therefore be corrected by Eqs. (8) before they are interpreted in terms of x, $dx/d\Theta$ via Eqs. (II-13).

We see from formulas (II-22) that all H_{33m} are of the same order as H_{33m_0} and $H_{31m} \sim 3H_{33m_0}$ except for those coefficients which happen to be small or vanishing either fortuitously or because of the deliberate arrangement of the sextupoles. The fractional corrections to the primed variables are therefore of the order of

$$\varepsilon = \frac{3A(2\rho)^{1/2}}{m - \ell\nu} , \qquad (9)$$

where $A = H_{33m_0}$. Thus at an amplitude $(2\rho)^{1/2} = X_0$, [Eq.(I-5)], near the separatrix, we expect fractional corrections of the order of

$$\varepsilon_0 = 0.3 \left(\frac{m_0 - 3\nu}{m - \ell\nu} \right) , \qquad (10)$$

from the term $H_{3\ell m}$. The error at the extraction septum will be larger by a factor X_e/X_{0i} in the notation of report I. We note from Eq. (10) that ε_0 is of the order of $|\nu - m_0/3|$, so that the error due to neglecting the correction (8) will be less than 1% provided we have eliminated those terms $H_{3\ell m}$ for which $|m - \ell\nu| < 1$. Since $\nu_0 = m_0/3$, we will have for the nearest unwanted resonances $|m - \ell\nu| \sim 1/3$, so that even they will contribute very little error.

We now calculate the sextupole contribution to H_4' by substituting from Eq. (7) in the higher order terms in Eq. (5):

$$H_4' = (2\rho')^2 \sum_{\ell,m,\ell',m'} \frac{3\ell' H_{3\ell m}\, H_{3\ell' m'}}{m' - \ell'\nu} \cos(\ell\gamma - m\Theta + \eta_{3\ell m}) \cdot$$

$$\cos(\ell'\gamma - m'\Theta + \eta_{3\ell' m'}) + \cdots , \qquad (11)$$

where the dots represent any octupole contributions which may be present. H_4' contains no terms which drive the resonance $\nu = m_0/3$, so that the only term of interest is $H_{4\ 0\ 0}$. The rest of the terms could be transferred to higher order by the method used above. The transformed Hamiltonian to fourth order is therefore

$$H' = \nu\rho' + H_{33m_0} (2\rho')^{3/2} \cos(3\gamma' - m_0\Theta + \eta_{33m_0})$$

$$+ H_{4\ 0\ 0} (2\rho')^2, \qquad (12)$$

where, from Eq. (11),

$$H_{4\ 0\ 0} = 3/2 \sum_{\ell=1,3} \sum_{m}{}' \frac{\ell H^2_{3\ell n}}{m - \ell\nu} + H^{oct}_{4\ 0\ 0} , \qquad (13)$$

where $H^{oct}_{4\ 0\ 0}$ is the octupole contribution, if any.

As an example, if there is a single sextupole, Eqs. (II-22) give

$$H_{3\ell m} = \frac{eR\beta^{3/2}F}{8\pi\ell M\gamma\omega} , \qquad (14)$$

where β is evaluated at the sextupole and F is its strength [defined by Eqs. (II-14)]. If there are no octupole contributions, Eqs. (13) and (14) give

$$H_{4\,0\,0} = \frac{\beta^3 e^2 R^2 F^2}{64\pi^2 M^2 \gamma^2 \omega^2} \left\{ \sum_{m=1}^{\infty} \left[\frac{3\nu - m_0}{m^2 - (3\nu - m_0)^2} + \frac{3(\nu - m_1)}{m^2 - (\nu - m_1)^2} \right] + \frac{1}{\nu - m_1} \right\} \doteq \frac{\pm 4.89 \beta^2 e^2 R^2 F^2}{64\pi^2 M^2 \gamma^2 \omega^2} , \quad (15)$$

where m_1 is the nearest integer to ν, we have taken $\nu = m_0/3$, and the result is positive if $\nu > m_1$, otherwise negative. If there are n sextupoles each of strength +F evenly spaced at homologous points around the machine, then $H_{3\ell m}$ vanishes unless m is a multiple of n, and in that case Eq. (14) holds with F replaced by nF. Clearly m_0 must be a multiple of n in order to drive the extraction resonance. In Eq. (15), F is replaced by nF, and m and m_1 must be multiples of n. The value of $H_{4\,0\,0}$ is then almost identical with the value given by the second line in Eq. (15) for a single sextupole of strength F if the nearest integer to ν is divisible by n; otherwise the number 4.89 is replaced by 5.27, 8.29, 10.0, 13.6, ... 1.9 k, if the nearest multiple of n differs from ν by 2/3, 4/3, 5/3, 7/3, ..., k/3. If n = 2n' is even, and the sextupoles strengths alternate signs, (±F), then $H_{3\ell m}$ vanishes unless m is an odd multiple of n', in which case F is again replaced by nF in formulas (14) and (15).

To reach a given driving amplitude $A = H_{33m_0}$ with n sextupoles requires n^{-1} times the sextupole strength F needed with one sextupole. Since $H_{4\,0\,0}$ is proportional to F^2 and is otherwise roughly independent of n, we reduce $H_{4\,0\,0}$ by roughly a factor n^{-2} with n appropriately arranged sextupoles.

In the n sextupoles are not at homologous points, or are not evenly spaced around the machine, the effective sextupole strength driving $\nu = m_0/3$ will still be roughly nF if the signs of the sextupoles are wisely chosen, but the cancellations will no longer occur in the sum over m in Eq. (13) which led to small numerators, and denominators proportional to m^2, in the sum in Eq. (15). The curly brackets in formula (15) will then be replaced by a numerical factor which can be as large as $4 \ln m_2$, where m_2 is a harmonic number above which $H_{3\ell m}$ becomes small. Since $m_2 \sim 2\pi/\Delta\Theta$, where $\Delta\Theta$ is the angular length of the sextupole, this factor may be as large as $4 \ln 10^4 \sim 40$ or ten times as large as with symmetrically placed sextupoles.